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ORIGINAL ARTICLE

Comments on "Steady open channel flows with curved streamlines: The Fawer approach revised"

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Abstract Steady open channel flow with curved streamlines cannot be treated with a hydraulic approach unless an extended Boussinesq-type energy equation is considered to account for the non-hydrostatic pressure distribution. The paper reviews approximate modeling of these flows by the notable theory of Carlos Fawer. This discussion intends to add to Fawer's model by comparing it with the Dressler curved-flow equations, not quoted in the original paper. Based on a comparative analysis of the present comments Fawer's theory is demonstrated to be relevant as compared to other formulations, supporting its importance, as highlighted in the original paper.

Keywords Hydraulics · Non-hydrostatic pressure · Open channel · Streamlined flow

The author discussed the approximate treatment of steady curved channel flows by presenting the original theory of Carlos Fawer [1]. The present discussion aims at widening the topic to steady open channel flows with curved streamlines by comparing the approach of Fawer [2], later considered by Matthew [3] and Hager [4], with the theory commonly proposed for these problems due to Dressler [5], not quoted in the paper, however.

Open channel flows are usually modelled by a hydraulic approach assuming uniform velocity and hydrostatic pressure distributions throughout the flow domain. Boussinesq, in his famous 'Essai sur la théorie des eaux courantes' published in 1877 [6], was the first to include streamline curvature effects in the momentum equation. One of the applications was the prediction of the solitary wave profile under well-defined flow conditions (i.e. pseudo-uniform flow in a rectangular channel of constant width and bottom slope [7,8]).

A first attempt to generalise the energy head equation for free surface flows must be attributed to Fawer [1]. The energy head for parallel-streamline flow reads

$$H = z + h + \frac{q^2}{2gh^2} \tag{1}$$

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where H = energy head, $z = z(x)$ channel bottom geometry, $h = h(x)$ free surface profile, q = discharge per unit width and g = gravitational acceleration. As stated in the paper, Fawer's approach was generalized by Matthew [3] to

$$H = z + h + \frac{q^2}{2gh^2}\left(1 + \frac{2hh'' - h'^2}{3} + hz'' - h'z' - z'^2\right) \tag{2}$$

with primes denoting ordinary differentiation with respect to the streamwise coordinate x. Evidently, Eqs. 1 and 2 are identical for horizontal, parallel-streamline flow.

Equation 2 may be considered a first order approximation of a more general relation. It may be demonstrated [4] that the magnitude of h'^2, hh'', $h'z'$, z'^2 and hz'' in the corrective term of Eq. 2 must be smaller than unity. Consequently, Fawer's and Matthew's approaches for the extended Boussinesq-type energy equation hold for flows with slightly-sloped and -curved streamlines. For flows with moderately-sloped and -curved streamlines (in particular the bottom and free surface streamlines) Eq. (2) may be generalized to [4]

$$H = z + h + \frac{q^2}{2gh^2}\exp\left(\frac{2hh'' - h'^2}{3} + hz'' - h'z' - z'^2\right) \tag{3}$$

in which terms in the bracket should not be larger than O(1). Evidently, the zeroth-order approximation of Eq. 3 corresponds to Eq. 1, while its first-order approximation is Eq. 2.

Dressler [4] presented inviscid open channel flow equations to account for the effect of bottom curvature by an asymptotic approximation of velocity and pressure in the irrotational fluid flow equations. For steady flow, the momentum and continuity equations read

$$\frac{CC'}{(1-\kappa N)^2} + \left[g\cos\theta + \frac{\kappa C^2}{(1-\kappa N)^3}\right]N' - \left[\kappa g\sin\theta - \frac{\kappa' C^2}{(1-\kappa N)^3}\right]N + g\sin\theta = 0 \tag{4}$$

$$\frac{CN'}{(1-\kappa N)^2} - \frac{\ln(1-\kappa N)}{\kappa(1-\kappa N)}C' + \frac{\kappa'}{\kappa^2}\left[\frac{\kappa N}{(1-\kappa N)^2} + \frac{\ln(1-\kappa N)}{(1-\kappa N)}\right]C = 0 \tag{5}$$

Further, $C = C(s)$ = bottom velocity, N = distance from channel bottom to free surface, measured orthogonally outwards from the bottom, θ and κ = slope and curvature of bottom profile, respectively, and primes denote ordinary differentiation with respect to the bottom streamline coordinate s. Let

$$q = -\frac{C}{\kappa}\ln(1-\kappa N) \tag{6}$$

Then the condition for steady flow ($\mathrm{d}q/\mathrm{d}s = 0$) reproduces immediately Dressler's Eq. (5). Further, with E = specific energy,

$$E = N\cos\theta + \frac{C^2}{2g(1-\kappa N)^2} \tag{7}$$

$$\frac{\mathrm{d}E}{\mathrm{d}x} = -\sin\theta \tag{8}$$

Equation (4) results if Eq. 7 is derived with respect to s and inserting Eq. 8 by noting that $\theta' = \kappa$. The above system of equations may be further simplified by eliminating C. Let $z = z(s)$ = bottom profile function, whence $z' = \sin\theta$. The simple result then is

$$H = z + N\cos\theta + \frac{q^2}{2gN^2}\left[\frac{\kappa N}{(1-\kappa N)\ln(1-\kappa N)}\right]^2 \tag{9}$$

which has the form of Eq. 3. However, there are two distinct differences in these formulations: (i) $N\cos\theta \neq h$ since the left hand-side refers to the free surface profile $N = N(x)$, while $h = h(x)$ is the pressure head profile [7]. The resulting deviations between the two may often be neglected for usual open channel flows; (ii) More significant is the absence of terms accounting for free surface profile variations (i.e. terms involving derivatives N' and N''). Consequently, Dressler's modified Eq. 9 applies *only* to open channel flows in which the effects of bottom curvature are significant, but where the free surface profile is nearly parallel to the bottom profile, i.e. $h' = h'' \approx 0$. Such cases are indeed realistic and may be found for flows over so-called ski-jumps ($\kappa > 0$) under supercritical flow conditions, and (partly) for spillway flow ($\kappa < 0$), as demonstrated by Sivakumaran et al. [9,10]. However, for arbitrarily sloping channels but *straight* bottom profiles, $\kappa = 0$, Dressler's modified Eq. 9 reduces directly to Eq. 1 for steady flow. This significant loss of information is an obvious disadvantage of the respective Dressler curved-flow equations as compared to Eq. 3, based on Fawer's theory.

Equation 9 may be further discussed if restricting considerations to slightly-sloped and -curved bottom geometries $z(s) \approx z(x)$, for which $O(\kappa^2 N^2) \rightarrow 0$. With $h \approx N\cos\theta$, and $\cos\theta = 1 - z'^2/2$, thus $1/N = (1 - z'^2/2)/h$, the first order approximation of Eq. 9 obtains

$$H = z + h + \frac{q^2}{2gN^2}(1 + \kappa N) = z + h + \frac{q^2}{2gh^2}\left(1 + hz'' - z'^2\right) \tag{10}$$

which is identical to Eq. 2, provided h is independent of x. As already stated, deviations between the two formulations are notable if the surface profile varies with the streamwise coordinate. Evidently, this is the usual case in open channel flows, for which the Dressler equations yield limited information as compared to Fawer's theory.

Another comparison between the two formulations examines the bottom curvature term Ω, involving the effects of z'' and κ, respectively, but neglecting the effect of z' ($\theta \approx 0$, i.e. extrema of the bottom geometry). Let

$$\Omega = \left[\frac{\kappa N}{(1 - \kappa N)\ln(1 - \kappa N)}\right]^2 \tag{11}$$

according to Eq. 9 while, in view of Eq. 10, the corresponding term of Eq. 3 is

$$\Omega = \exp(\kappa N) \tag{12}$$

Figure 1 compares these two Ω with the "exact" value according to [4]. Ω according to Eq. 11 lies always above the others, and respective deviations are significant except for

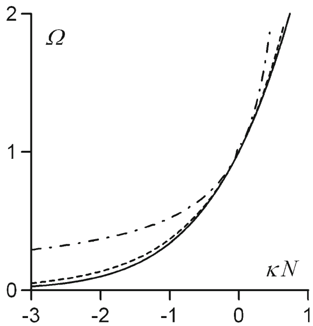

Fig. 1 Comparison of $\Omega = \Omega(\kappa N)$ according to (— - —) Eq. 11, (- - -) Eq. 12 and (—) exact solution according to [4]

$-1 \leq \kappa N \leq +0.5$. Dressler proposed this range as approximate applicability domain of his model Eq. 5.

The comparative analysis presented above supports the fundamental validity of Fawer's original theory, as compared to other limited formulations. Note that Dressler [5] wrote in his paper "*A few years after Saint-Venant's work Boussinesq derived additional equations representing higher-order corrections to the shallow-flow equations in order to include some curvature effects in the particle velocities, not present in Saint-Venant's equations. Boussinesq's method, however, was not systematic or rigorous, and his results are primarily of historical interest now*". However, both the present comment and the original paper support the rigorous mathematical development on which are based both Boussinesq's and Fawer's theories.

I would of course be interested in the Author's comments. Note that both Robert Dressler (1920–1999) and his colleague Vujica Yevjevich (1913–2006) with whom another paper on this problem was published have passed away.

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